Introduction

Mathematics teaching has always been a curious blend of the old and the new. This is particularly true as the use of technology becomes more commonplace in school classrooms. When teachers and students revisit traditional topics with technology, they are afforded opportunities to connect mathematical ideas in powerful, previously unimagined ways. The National Council of Teachers of Mathematics (NCTM) captures the importance of connections clearly in its *Principles and Standards of School Mathematics* (2000).

The notion that mathematical ideas are connected should permeate the school mathematics experience at all levels. As students progress through their school mathematics experience, their ability to see the same mathematical structure in seemingly different settings should increase (p. 64).

In the paragraphs that follow, we describe a recent technology-oriented investigation that led us to surprising connections among quadratics, mathematical envelopes, tangent lines, and tangent parabolas. In particular, our work with the *TI-Nspire* computer algebra system (CAS) enabled our students to generate conjectures and test hypotheses in ways not possible with pencil and paper alone while providing new insights into traditional, skill-oriented topics.

Completing the Square with Tiles: An Overview

Recently, while attending a local teaching conference focusing on the use of interactive whiteboards (IWB), our interest was piqued by the speaker’s discussion of a seemingly routine
topic, namely *completing the square*. The talk began innocently enough, with the speaker demonstrating the drawing capabilities of IWB software. As he represented algebraic expressions with virtual *Algebra Tiles* (Bruner, 1964; Dienes, 1960; ETA/Cuisenaire, 2008; Picciotto, 2008), the speaker used tiles of three different dimensions - with areas of 1, $x$, and $x^2$ square units (Corn, 2004). The shapes, illustrated in Figure 1, were familiar to many audience members.

![Figure 1: Three basic types of tiles used to represent algebra terms within IWB software.](image1)

The speaker combined tiles together to represent more complicated algebraic expressions. For instance, the trinomial $x^2 + 6x + 1$ was represented as shown in Figure 2.

![Figure 2: The algebraic expression $x^2 + 6x + 1$ as represented with virtual tiles within the IWB environment.](image2)

By manipulating the tiles within the IWB environment, the presenter demonstrated several completing the square examples such as the one illustrated in Figure 3.
The arrangement of tiles in Figure 3 (left) suggests that 8 units are needed to "complete the square" for the expression \(x^2 + 6x + 1\). The expression that results from this addition, namely \(x^2 + 6x + 9\), is a perfect square trinomial, expressible as the square of a binomial, \((x+3)^2\). This is suggested in Figure 3 (right).

The teacher shared several more examples of this sort, each time completing the square by splitting the \(x\) tiles into two piles with the same number of pieces - placing one pile in a horizontal "stack" below the \(x^2\) piece and the other pile in a vertical "stack" to the right of the \(x^2\) piece. While discussing pedagogical advantages and limitations of such an approach, the presenter made an off-handed comment that was rather intriguing to us. Specifically, he noted that tiles weren't helpful when completing the square with trinomials containing odd linear coefficients (such as \(x^2+5x+1\)) because \textit{odd numbers of }\(x\) \textit{tiles can't be split into two piles with the same number of pieces.}

Although no one questioned this claim, we sat through the remainder of the session...
wondering if this were a true statement. Since we happened to have a laptop with presentation software handy, we modeled the expression with tiles and constructed squares by *adding pieces.* Rather than splitting the $x$ blocks into two equal piles, we added blocks of *various dimensions* onto our initial construction in order to *complete* the square. Before the end of the presentation, we had generated a family of expressions that could be added to $x^2+5x+1$ to "complete squares."

Several examples of "completed squares" are shown in Figure 4.

![Figure 4](image)

**Figure 4:** Given the expression $x^2 +5x + 1$, one may "complete a square" in various ways. For instance, a square may be completed by adding either $x+8$ (Left) or $3x+15$ (Right).

In the next section, we explore unexpected mathematical connections fostered by our new interpretation of "completing a square." In particular, we discuss the following:

1. The family of linear functions that complete squares for various trinomials (i.e. "square completers") and the relationship of these functions to the original (i.e. "seed") trinomial;

2. Connections between "square completer" families and the traditional completing the square algorithm.

Technology plays a central role throughout our exploration, informed by concrete. For instance,
interactive white boards and virtual Algebra Tiles motivated our initial questions about alternative conceptions of completing the square. Subsequent explorations involving more sophisticated mathematical simulation and data analysis are fostered using TI-Nspire CAS and dynamic geometry software (DGS).

**A New Look at an Old Problem: Families of "Square Completers"**

Consider our previous example involving $x^2 + 5x + 1$. While students are typically taught that exactly one expression will "complete the square" for such a trinomial, in fact, any one of an entire family of linear expressions may be added to generate perfect square trinomials. For instance, in Figure 4 we illustrated that adding $x+8$ to $x^2 + 5x + 1$ generates the perfect square trinomial $x^2+6x+9 = (x+3)^2$. Likewise, adding $3x+15$ yields $x^2+8x+16 = (x+4)^2$. In fact, an infinite number of different expressions may be added to $x^2+5x+1$ to generate perfect square trinomials. We define the $n$th "square completer" of trinomial $p(x)$ as the expression one adds to $p(x)$ to yield perfect square trinomial $(x+n)^2$. Table 1 illustrates family members for $p(x) = x^2 + 5x + 1$ for various $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$th square completer</th>
<th>Sum</th>
<th>Factored Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$x+8$</td>
<td>$(x^2+5x+1)+(x+8) = x^2+6x+9$</td>
<td>$(x+3)^2$</td>
</tr>
<tr>
<td>4</td>
<td>$3x+15$</td>
<td>$(x^2+5x+1)+(3x+15) = x^2+8x+16$</td>
<td>$(x+4)^2$</td>
</tr>
<tr>
<td>5</td>
<td>$5x+24$</td>
<td>$(x^2+5x+1)+(5x+24) = x^2+10x+25$</td>
<td>$(x+5)^2$</td>
</tr>
<tr>
<td>6</td>
<td>$7x+35$</td>
<td>$(x^2+5x+1)+(7x+35) = x^2+12x+36$</td>
<td>$(x+6)^2$</td>
</tr>
<tr>
<td>100</td>
<td>$195x+9999$</td>
<td>$(x^2+5x+1)+(195x+9999)$</td>
<td>$(x+100)^2$</td>
</tr>
</tbody>
</table>

Note that the second differences from entries in 2nd column are constant, suggesting a quadratic
relationship between $n$ and the $n$th square completer. Later in this paper, we discuss a CAS-assisted derivation of a general formula for the $n$th square completer for arbitrary $p(x)$.

**Initial Investigations with CAS**

To gain a more thorough understanding of the family of square completers, we construct a multi-page *TI-Nspire* document to automatically generate family members for given trinomial $p(x)$. On the first page of the document, students use sliders to provide coefficients of the seed trinomial, $ax^2 + bx + c$. On the second page, algebraic expressions for various family members are generated within a CAS-enhanced spreadsheet. Screens from the first two pages of the *Nspire* document are highlighted in Figure 5.

![Figure 5: (Left) Parameters a, b, and c are entered with sliders. A graph of selected "square completers" is generated from these parameters; (Right) A CAS-enhanced spreadsheet calculates algebraic expressions for square completers (column D) using linked slider values.](image)

As the graph in Figure 5 (left) suggests, the envelope formed by family members appears to be quadratic. To explore this conjecture more rigorously, intersection points among family members are constructed, then a quadratic "fit" function is generated for these ordered pairs. This approach is illustrated in Figure 6, left. The fit calculations associated with the envelope of
family members for \( x^2 + 5x + 1 \) is shown in Figure 6 (right).

![Figure 6: (Left) The white dots depict intersections of \( n \)th and \((n+1)\)st square completer graphs. The bold curve is generated by fitting a quadratic function to these intersections; (Right) Calculations associated with the quadratic "fit" curve.](image)

As students manipulate values of \( a \), \( b \), and \( c \) with sliders, the envelope and associated quadratic regression are updated dynamically. This enables students to look for connections between various seed polynomials and corresponding envelopes of square completers such as those provided in Table 2. Without technology, such an exploration would be inaccessible to secondary school students due to the time required to generate such graphs by hand.

**Table 2: Seed Polynomials and Corresponding Quadratic Envelope Equations.**

<table>
<thead>
<tr>
<th>Seed Polynomial</th>
<th>Envelope of Square Completers</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2+5x+1 )</td>
<td>(-x^2-5x-0.75)</td>
</tr>
<tr>
<td>( x^2+5x+2 )</td>
<td>(-x^2-5x-1.75)</td>
</tr>
<tr>
<td>( x^2+5x+3 )</td>
<td>(-x^2-5x-2.75)</td>
</tr>
<tr>
<td>( x^2+6x+1 )</td>
<td>(-x^2-6x-0.75)</td>
</tr>
<tr>
<td>( x^2+6x+2 )</td>
<td>(-x^2-6x-1.75)</td>
</tr>
<tr>
<td>( x^2+6x+3 )</td>
<td>(-x^2-6x-2.75)</td>
</tr>
<tr>
<td>( x^2+9x+9 )</td>
<td>(-x^2-9x-8.75)</td>
</tr>
</tbody>
</table>

Clearly, the entries in Table 2 suggest a relationship between the equations of quadratic envelopes.
and seed polynomials - namely, seed \( p(x) = x^2 + bx + c \) appears to have envelope \(-x^2 - bx - c + 0.25\). While this conjecture may not hold in general, the cases warrant further investigation.

**Connections with Tangent Curves**

Because the sign of the coefficients of the seed and envelope appear to be opposites, we modify our *Nspire* document to graph the *opposites* of "square completers" of \( p(x) \), defining \( q_n(x) \) as the opposite of the \( n \)th square completer for a given \( p(x) \). Explicitly,

\[
q_n(x) = p(x) - (x + n)^2
\]

The modified *Nspire* document simultaneously graphs \( q_n(x) \) for various \( n \) and \( p(x) \). An example graph is depicted in Figure 7 for \( p(x) = x^2 + 5x + 1 \) and \( q_n(x) \) with \(-20 < n < 20\).

![Figure 7: The graph of \( p(x) \) and \( q_n(x) \) with \(-20 < n < 20\).](image)

By inspection, students conjecture that the graph of \( q_n(x) \) is tangent to \( p(x) \) for all \( n \). To determine if such a hypothesis is true, we first construct a formula for \( q_n(x) \) for \( p(x) = x^2 + bx + c \).

\[
q_n(x) = p(x) - (x + n)^2
\]

by definition

\[
= x^2 + bx + c - (x^2 + 2nx + n^2)
\]

by substitution

\[
= (b-2n)x + (c - n^2)
\]

by algebraic simplification

Similar calculations can be performed in a step-by-step fashion with CAS. Such an approach is
highlighted in Figure 8 with a screenshot from TI-Nspire CAS.

\[
p(x) := x^2 + bx + c \\
q(n, x) := p(x) - (x + n)^2 \\
q(n, x) = (b - 2n)x + c - n^2
\]

**Figure 8:** Calculation of \( q_n(x) \) as implemented with TI-Nspire CAS.

Insight into the general formula of \( q_n(x) \) is provided through analysis of a concrete representation of \((x-n)^2 - (x^2+bx+c)\) with Algebra Tiles as that illustrated in Figure 9.

**Figure 9:** An representation of \( q_n(x) = (x^2+bx+c) - (x+n)^2 \) with Algebra Tiles. Together, the grey and white regions (i.e. \((2n-b)x + n^2-c\) represent the \(n\)th square completer of \(x^2+bx+c\) (i.e. \(-q_n(x)\)).
With $q_n(x)$ defined as $-(2n-b)x-(n^2-c)$ for arbitrary $p(x)=x^2+bx+c$, students use CAS to verify that the graph of $q_n(x)$ is tangent to $p(x)$ for all $n$.

**CAS-based Proof.** First, we define $p(x)$ as the general trinomial $x^2+bx+c$ and $q_n(x)$ by definition as $-(2n-b)x-(n^2-c)$. Solving the equation $p(x)=q_n(x)$, we determine intersection points of $x^2+bx+c$ and $q_n(x)$, the $n$th member of the square completers. Solutions to the equation, as calculated with *TI-Nspire* CAS, are shown in Figure 10.

![Figure 10: Determining points of intersection for $p(x)$ and $q_n(x)$ with CAS.](image)

From Figure 10, we see that there is precisely one solution to the equation, namely $x=-n$. This strongly suggests that every member of the square completer family $q_n(x)$ is tangent to $p(x)$ at $x=-n$. Establishing with certainty that $q_n(x)$ is tangent to $p(x)$ requires little more than elementary calculus. Noticing that the slope of $q_n(x)$ is $-(2n-b)$, this is precisely the value of $p'(x)=2x+b$ evaluated at $x=-n$. Hence, $q_n(x)$ and the tangent line to $p(x)$ at $x=-n$ are parallel. However, $q_n(x)$ and $p(x)$ have only one point in common. Therefore $q_n(x)$ is the tangent line to $p(x)$ at $x=-n$. 
Graphical Interpretation of Square Completers

How do p(x), q_x(n) and the completed square (x+n)^2 fit together graphically? Consider p(x)=x^2+5x+1 and the specific square completers q_-1(x) and q_2(x) as shown in figure 11. We have established that each being a member of q_n(x), that each is tangent to p(x) at x=1 and x=-2, respectively. Moreover, each member of q_n(x) leads to a completed square trinomial whose vertex is located on the x-axis directly above (or below) the point of tangency. By completing the square in this manner (adding on the appropriate Algebra Tiles), we effectively translate p(x) so that the vertex is vertically aligned with the point of tangency and on the x-axis.

Completing the Square Revisited

At this juncture, it is instructive to connect our new observation back to more traditional notions of completing the square (i.e. taking the half the linear coefficient, squaring, and adding). In the traditionally understood meaning of completing the square, students add nothing but unit blocks to the initial trinomial - they don’t add x-blocks. Algebraically, this implies that q_n(x) = k, a constant function. Hence, when completing the square in the conventional sense, the
coefficient of the linear term of \( q_n(x) \) is zero - in other words, \(-(2n-b) = 0\) implying \( n = b/2\).

Recalling that \( q_n(x) \) is tangent to \( p(x) \) at \( x=-n \), this implies that in traditional completing the square situations, \( q_n(x) \) is a horizontal line tangent to \( p(x) \) at the vertex of the trinomial.

This leads to a novel interpretation of completing the square. Geometrically speaking, by completing the square, one translates \( p(x) \) vertically so that its vertex lies on the \( x \)-axis. Consider again our initial example, namely \( p(x) = x^2 + 5x + 1 \). When students complete the square for \( p(x) \), they perform steps similar to the following:

\[
\begin{align*}
 p(x) &= x^2 + 5x + 1 \\
 p(x) + \left(\frac{5}{2}\right)^2 - 1 &= (x^2 + 5x + 1) + \left(\frac{5}{2}\right)^2 - 1 \\
 p(x) + 5.25 &= x^2 + 5x + 6.25 \\
 p(x) + 5.25 &= (x + 2.5)^2 \\
 p(x) &= (x + 2.5)^2 - 5.25
\end{align*}
\]

In step 4 of the above calculations, we see that the seed function \( p(x) \) is translated vertically +5.25 units. In particular, the vertex of \( p(x) \) is translated from \((-5/2, -5.25)\) to \((-5/2, 0)\). This idea may be represented graphically by plotting \( p(x) \) and \( q_{-2.5}(x) \) simultaneously with TI-Nspire CAS as shown in Figure 12.

![Figure 12: Graphical interpretation of completing the square with simultaneous plots of \( p(x) \) and \( q_{-2.5}(x) \).](image)
Recalling prior discussions of the square completers $q_{-1}(x)$ and $q_{2}(x)$, students view the traditional method of completing the square as a special case of a much larger "completing the square" theory.

**Four Suggestions for Future Research**

There are many fruitful paths to explore when we combine the concrete visual clues of Algebra Tiles with the rich technology applications found in the TI-Nspire CAS. The following examples offer interested readers a glimpse of the ways a technologically enhanced algebra tile view of polynomials lead students to previously unforeseen connections. Below, we briefly describe four investigations that are readily accessible to students in first-year algebra through Pre-Calculus.

**Quadratics With Leading Coefficients Other Than 1**

Consider $p(x) = 2x^2 + 5x + 1$. The traditional algorithm for this case would have students first divide through by 2 (the leading coefficient) to make the leading coefficient 1, then proceed as the traditional algorithm dictates (i.e. take half the linear coefficient, square it, and add). Instead, we look for opportunities to *add on tiles* to complete a square. As we have established, many combinations of Algebra tiles would suffice to complete a square; for instance, adding $2x^2 + 3x + 2$ would work, as would adding on $2x^2 + 7x + 14$. Graphing $p(x)$ along with the family of the opposites of all possible square completers (i.e. $q_n(x)$) produces the graph in Figure 13.
Completing the Square for Linear Equations

Begin with a single $x$-block and 3 unit blocks modeling $x+3$. What algebra tiles would one need to add in order to complete the square? One could add $x^2+3x+1$ or even $x^2+9x+22$. Either way, if you examine the family of square completers, the result is a family of parabolas tangent to the line $y = x + 3$, as shown in Figure 14.

Completing Cubes

Figure 15 depicts 3D Algebra Blocks modeling $p(x)=x^3+2x^2+x$. What would be required to complete the cube? In the same manner as above, we could add $x^2+2x+1$, or
4x^2+15x+8, or 7x^2+90x+27 to name just a few cube completers. Graphing the opposite of these yields a family of parabolas in which some are tangent, but in which each member intersects the cubic p(x) in only one point.

![Figure 15: (Left) Algebra Blocks modeling p(x) = x^3+2x^2+x; (Right) A family of parabolas - some tangent, some not - each intersecting the cubic p(x) in one point.](image)

**Squaring the Circle**

Experienced *Algebra Tile* users will recognize both x and y blocks. Many of these users also are comfortable with various method for modeling negative integers. These methods can be used to model quadratics that represent conics. For instance, consider the collection of x^2+y^2-4 algebra tiles used to model the circle x^2+y^2=4. In general, to complete the square (x+y+n)^2, you *could* add 2xy+2x+2y+5, or 2xy+4x+4y+8, or 2xy+16x+16y+68. Graphing the circle and the family of square completers for the circle produces a family of hyperbolas tangent to the circle, producing graphs such as that depicted in Figure 16.
Summary

In the previous discussion, alternative interpretations of completing the square were explored along with explorations of mathematical implications of such interpretations. Throughout our investigations, technology has played a central role, informed by concrete Algebra Tiles.

From the generation of initial questions with Interactive White Boards (IWB) to the construction and testing of hypotheses with students using TI-Nspire CAS and dynamic geometry software (DGS), technology has made various facets of our investigation accessible to a secondary school audience. In particular, the tools provide our students with a means of considering abstract mathematical objects such as the manipulation of symbolic polynomials more concretely as geometric objects. Moreover, the use of technology afforded both us and our students with opportunities to connect mathematical ideas in powerful, previously unimagined ways. The process of completing the square is transformed from a series of algebraic steps to be memorized,
executed, and quickly forgotten to a geometric process that reveals the vertex of a parabola through geometric translation.

References


Corn, J. (Nov. 13, 2004). *Completing the Square: Virtual Algebra Tiles, Smart Boards, and SMG*. Presentation at Teachers Teaching with Technology Regional Conference, Cleveland, OH.


